

Efficient multiplication

Introducing a new algorithm

Efficient multiplication

- Say you want to multiply two large numbers
- For now, let us assume they have an equal number of digits
 - Here “digit” is a loose concept
 - A digit could be
 - a single decimal digit
 - one hundred decimal digits
 - a byte's worth of binary digits
 - a megabyte's worth of binary digits
 - The actual unit used will depend on the implementation

Efficient multiplication

- Suppose the two numbers are

34 567 891 234

and

87 456 123 876

- Since these numbers contain 11 digits each, the answer will require $11+11=22$ digits at most
- Create a (upside down) pyramid - we'll use the pyramid to multiply
- Write these two numbers above the pyramid's base
 - Write each digit so that it occupies two digits' worth of space
 - Thus each number spans 22 digits and also spans the space we'll use for the answer

Efficient multiplication

- Thus the base is created as shown below, with each digit taking up two digits' worth of space

a's digits	3	4	5	6	7	8	9	1	2	3	4
b's digits	8	7	4	5	6	1	2	3	8	7	6

- Next we multiply each of the digits in the first row with those in the second row and write the answer into the “base” of the pyramid

– Thus $3 \times 8 = 24$ for example

- Write each two digit answer into its own block
- Write each digit into its own cell

a's digits	3	4	5	6	7	8	9	1	2	3	4											
b's digits	8	7	4	5	6	1	2	3	8	7	6											
base = a x b	2	4	2	8	2	0	3	0	4	2	0	8	1	8	0	3	1	6	2	1	2	4

Efficient multiplication

- Now complete the pyramid row by row
 - Each row contains blocks
 - Each block contains two double digit numbers
 - The first number is $a_i \times b_j$
 - The second number is $a_j \times b_i$
 - Here a comes from the top row or first number's digit
 - Likewise b comes from the bottom row or second number's digits
 - The difference between i and j remains constant for each row
 - The difference increases each time we move to the next row
 - The example below shows the first row of blocks in the pyramid

a's digits	3	4	5	6	7	8	9	1	2	3	4
b's digits	8	7	4	5	6	1	2	3	8	7	6

base = a x b	2	4	2	8	2	0	3	0	4	2	0	8	1	8	0	3	1	6	2	1	2	4
3 x 7 = 21	2	1	1	6	2	5	3	6	0	7	1	6	2	7	0	8	1	4	1	8		
8 x 4 = 32	3	2	3	5	2	4	3	5	4	8	0	9	0	2	0	6	2	4	2	8		

Efficient multiplication

- The first row of blocks in the pyramid again

a's digits	3	4	5	6	7	8	9	1	2	3	4
b's digits	8	7	4	5	6	1	2	3	8	7	6

base = a x b	2	4	2	8	2	0	3	0	4	2	0	8	1	8	0	3	1	6	2	1	2	4
3 x 7 = 21	2	1	1	6	2	5	3	6	0	7	1	6	2	7	0	8	1	4	1	8		
8 x 4 = 32	3	2	3	5	2	4	3	5	4	8	0	9	0	2	0	6	2	4	2	8		

- Note that the blocks contains two two digit numbers
 - Each number is the product of digits from a and b spaced 1 apart
 - $3 \times 7 = 21$ as the first number in the first block
 - $8 \times 4 = 32$ as the second number in the first block
 and so on for the whole row

Efficient multiplication

- The next row is constructed just as before
- Only this time we use numbers from a and b spaced 2 apart

a's digits	3	4	5	6	7	8	9	1	2	3	4
b's digits	8	7	4	5	6	1	2	3	8	7	6

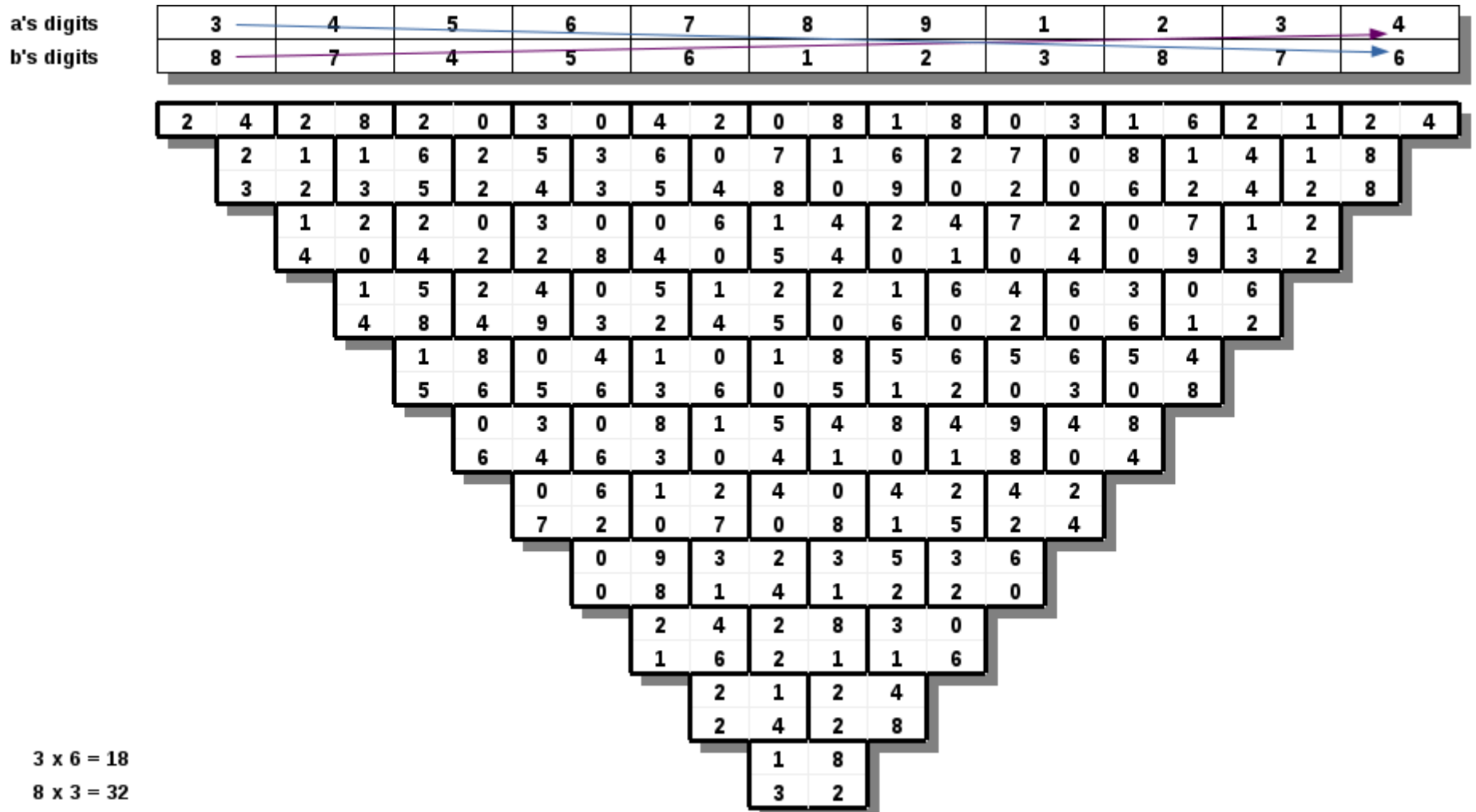
2	4	2	8	2	0	3	0	4	2	0	8	1	8	0	3	1	6	2	1	2	4
2	1	1	6	2	5	3	6	0	7	1	6	2	7	0	8	1	4	1	8		
3	2	3	5	2	4	3	5	4	8	0	9	0	2	0	6	2	4	2	8		
1	2	2	0	3	0	0	6	1	4	2	4	7	2	0	7	1	2				
4	0	4	2	2	8	4	0	5	4	0	1	0	4	0	9	3	2				

$3 \times 4 = 12$
 $8 \times 5 = 40$

- Thus the first block contains $3 \times 4 = 12$ and $8 \times 5 = 40$ and so on for the other blocks

Efficient multiplication

- We continue in this fashion until the pyramid is fully constructed as shown below
 - Note how we calculate the final row's block as the product of the outer digits of a and b



Efficient multiplication

- When the pyramid is completed, we sum the digits in each column
- Each column contains only single digit numbers
 - The sum will typically be larger than a single digit
 - We treat the carry as a normal carry and add it to the next column or digit and so on
 - It is easy if we write the carry on top of the next digit in the answer space
 - This is shown on the next slide

Efficient multiplication

a's digits
b's digits

3	4	5	6	7	8	9	1	2	3	4
8	7	4	5	6	1	2	3	8	7	6

2	4	2	8	2	0	3	0	4	2	0	8	1	8	0	3	1	6	2	1	2	4
	2	1	1	6	2	5	3	6	0	7	1	6	2	7	0	8	1	4	1	8	
		3	2	3	5	2	4	3	5	4	8	0	9	0	2	0	6	2	4	2	8
			1	2	2	0	3	0	0	6	1	4	2	4	7	2	0	7	1	2	
				4	0	4	2	2	8	4	0	5	4	0	1	0	4	0	9	3	2
					1	5	2	4	0	5	1	2	2	1	6	4	6	3	0	6	
						4	8	4	9	3	2	4	5	0	6	0	2	0	6	1	2
							1	8	0	4	1	0	1	8	5	6	5	6	5	4	
								5	6	5	6	3	6	0	5	1	2	0	3	0	8
									0	3	0	8	1	5	4	8	4	9	4	8	
										6	4	6	3	0	4	1	0	1	8	0	4
											0	6	1	2	4	0	4	2	4	2	
												7	2	0	7	0	8	1	5	2	4
													0	9	3	2	3	5	3	6	
														0	8	1	4	1	2	2	0
															2	4	2	8	3	0	
																1	6	2	1	1	6
																	2	1	2	4	
																		2	4	2	8
																			1	8	
																				3	2

Carry
Sum

1	1	2	4	3	5	4	6	5	6	7	7	5	5	3	4	4	2	0	1	0	
3	0	2	3	1	7	3	7	7	7	8	9	2	7	9	8	5	0	2	9	8	4

2+8+8=18, write 8 here
Carry is 1, written here
Carry of 1 is added to next digit

Efficient multiplication

- There we have it 34 567 891 234 x 87 456 123 876
= 3 023 173 777 892 798 502 984

a's digits
b's digits

3	4	5	6	7	8	9	1	2	3	4
8	7	4	5	6	1	2	3	8	7	6

2	4	2	8	2	0	3	0	4	2	0	8	1	8	0	3	1	6	2	1	2	4	
	2	1	1	6	2	5	3	6	0	7	1	6	2	7	0	8	1	4	1	8		
		3	2	3	5	2	4	3	5	4	8	0	9	0	2	0	6	2	4	2	8	
			1	2	2	0	3	0	0	6	1	4	2	4	7	2	0	7	1	2		
				4	0	4	2	2	8	4	0	5	4	0	1	0	4	0	9	3	2	
					1	5	2	4	0	5	1	2	2	1	6	4	6	3	0	6		
						4	8	4	9	3	2	4	5	0	6	0	2	0	6	1	2	
							1	8	0	4	1	0	1	8	5	6	5	6	5	4		
								5	6	5	6	3	6	0	5	1	2	0	3	0	8	
									0	3	0	8	1	5	4	8	4	9	4	8		
										6	4	6	3	0	4	1	0	1	8	0	4	
											0	6	1	2	4	0	4	2	4	2		
												7	2	0	7	0	8	1	5	2	4	
													0	9	3	2	3	5	3	6		
														0	8	1	4	1	2	2	0	
																2	4	2	8	3	0	
																	1	6	2	1	1	6
																		2	1	2	4	
																			2	4	2	8
																				1	8	
																					3	2

Carry
Sum

1	1	2	4	3	5	4	6	5	6	7	7	5	5	3	4	4	2	0	1	0	
3	0	2	3	1	7	3	7	7	7	8	9	2	7	9	8	5	0	2	9	8	4

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for more information or send comments to

multiply@ferrasolutions.com